Non-topological solitons in field theories with kinetic self-coupling

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(Dated: February 1, 2008)

We investigate some fundamental features of a class of non-linear relativistic lagrangian field theories with kinetic self-coupling. We focus our attention upon theories admitting static, spherically symmetric solutions in three space dimensions which are finite-energy and stable. We determine general conditions for the existence and stability of these non-topological soliton solutions. In particular, we perform a linear stability analysis that goes beyond the usual Derrick-like criteria. On the basis of these considerations we obtain a complete characterization of the soliton-supporting members of the aforementioned class of non-linear field theories. We then classify the family of soliton-supporting theories according to the central and asymptotic behaviors of the soliton field, and provide illustrative explicit examples of models belonging to each of the corresponding sub-families. In the present work we restrict most of our considerations to one and many-components scalar models. We show that in these cases the finite-energy static spherically symmetric solutions are stable against charge-preserving perturbations, provided that the vacuum energy of the model vanishes and the energy density is positive definite. We also discuss briefly the extension of the present approach to models involving other types of fields, but a detailed study of this more general scenario will be addressed in a separate publication.

PACS numbers: 05.45.Yv, 11.10.-z, 11.10.Lm, 11.27.+d

The non-linear Born-Infeld (BI) field theory was originally introduced to remove the divergence of the electron's self-energy in classical electrodynamics [1]. The BI procedure defines a new lagrangian for the electromagnetic field, as a given function of the two field invariants, which reduces to the Maxwell lagrangian in the low-energy limit and exhibits central-field static solutions which are finite-energy and stable. This procedure has been extended and widely used in many different contexts in Theoretical Physics. As examples of these extensions we mention: a) The generalization of the BI procedure to non-abelian gauge fields and to higher space dimensions, suggested by some results in the context of M theory five-brane [2]. b) The search for self-gravitating soliton solutions of Einstein equations [3]. c) The description of dark energy in the context of Cosmology, as a gauge field governed by a BI-like action, or as scalar fields with several kinds of derivative self-couplings [4]. d) The elaboration of chiral models with soliton solutions for the phenomenological description of the nucleon structure, as an alternative to the usual Skyrme model with a stabilizing quartic term in the lagrangian [5]. Almost all of these generalizations introduce modified lagrangians which are obtained from the initial ones using the same BI prescription. Besides historical reasons, this choice is supported by some nice properties of the original and extended BI actions, as the existence of finite-energy (electrostatic and dyon-like) solutions [6], the *exceptional* properties of wave propagation (in the electromagnetic original version [7]) or the improvement of the stability behavior. It is clear that the physical motivations for studying field theories inspired on the BI approach are many-fold. Consequently, it is of considerable interest to establish under what conditions these field theories admit stable particle-like solutions. The aim of the present work is to determine these conditions and to use them to characterize the set of models supporting the aforementioned type of solutions. As we shall see, the alluded conditions define exhaustively this set of models and allow for their explicit determination. This provides a wide panoply of supplementary tools in order to address the aforementioned problems and others.

We have first approached these questions in the simpler case of scalar fields ($\phi(x^\mu)$), with lagrangian densities defined as *arbitrary functions* of the kinetic term ($X = \partial_\mu\phi.\partial^\mu\phi$). This is the natural restriction to the scalar case of the more general problem outlined for electromagnetic fields, where the general lagrangians can be defined as arbitrary functions of the field invariants. Moreover, the analysis of this problem and its extension to the case of many-components scalar fields, are basic steps for the later generalization to the case of abelian and non-abelian gauge fields. However the results for gauge models will be presented in separate publications [10], [11].

The search for this kind of scalar models exhibiting static soliton solutions in three space dimensions, by circumventing the hypothesis of the Derrick theorem [8], has been already partially discussed in an old paper [9]. Here we go beyond these results by establishing the general conditions for the existence of such solutions and performing a general analysis of their linear stability, beyond the Derrick (*necessary*) criterion. Most of this analysis can be easily generalized to other dimensions.

Let us start with the lagrangian density

$$L = f(\partial_\mu\phi.\partial^\mu\phi), \tag{1}$$

where the function $f(X)$ is assumed to be *continuous and derivable* in the domain of definition (Ω). For the purposes of the present analysis we shall call "class-1" field theories the models (1) for which $f(X)$ is defined and regular everywhere ($\Omega \equiv \Re$) and "class-2" field theories those with $\Omega \subset \Re, 0 \in \Omega$ and Ω open and connected. For obvious physical reasons other models are excluded. In all cases the associated field equations take the form of a local conservation law (in what follows we denote as $\dot{f}(X)$ and $\ddot{f}(X)$ the first and second derivatives of $f(X)$, respectively)

$$\partial_\mu \left[\dot{f}(X) \partial^\mu \phi \right] = 0. \tag{2}$$

The energy density obtained from the canonical energy-momentum tensor is

$$\rho = 2\dot{f}(X) \left(\frac{\partial \phi}{\partial t} \right)^2 - f(X). \tag{3}$$

We require this energy density to vanish in vacuum and to be positive definite everywhere. This imposes the following supplementary restrictions on the Lagrangian density:

$$f(0) = 0 \;\; ; \;\; \dot{f}(X) \geq 0 \;\; (\forall X) \;\; ; \;\; f(X) \leq 0 \;\; (\forall X \leq 0)$$
$$\tfrac{d}{dX}\left(\tfrac{f^2(X)}{X}\right) \geq 0 \;\; (\forall X > 0). \tag{4}$$

We shall call "admissible" the field models satisfying these requirements. For static spherically symmetric (SSS) solutions $\phi(r)$, Eq.(2) can be integrated once, leading to

$$r^2 \phi^{'} \dot{f}(-\phi^{'2}) = \Lambda, \tag{5}$$

where $\phi^{'} = d\phi/dr$, and Λ is the integration constant. From this equation and the conditions (4) we see that $\phi^{'}(r)$ (if unique) must be a monotonic function. Strictly speaking Eq.(5) determines the field strength for $r > 0$ only. If we substitute the solutions of this equation in (2) we obtain a Dirac δ distribution of weight $4\pi\Lambda$. We can then identify this parameter as the "source point charge" associated with the scalar SSS solution (alternatively, if $\phi^{'}(r)$ is asymptotically coulombian, Λ can be interpreted as the total scalar charge, continuously distributed in space with a density $\vec{\nabla}^2\phi$, in analogy with the definitions in the non-linear BI models [1]).

The total energy of the SSS solutions obtained by integration of (3) reads

$$\varepsilon(\Lambda) = -4\pi \int_0^\infty r^2 f(-\phi^{'2}(r,\Lambda)) dr = \Lambda^{3/2} \varepsilon(\Lambda = 1), \tag{6}$$

where the last equality is a consequence of the invariance of the solutions of Eq.(2) under the scale transformations $\phi(\vec{r}, t) \to \lambda^{-1}\phi(\lambda\vec{r}, \lambda t)$. The convergence of this integral depends on the behavior of the SSS field strength at the origin and at $r \to \infty$. If we assume a power law expression ($\phi^{'}(r) \sim r^q$) in both cases [14], it follows from Eq.(5) that the integrand in (6) behaves as $r^2 f(-\phi^{'2}(r)) \sim \phi^{'}(r) \sim r^q$ and thus the convergence of the energy integral requires $q > -1$ when $r \sim 0$ and $q < -1$ as $r \to \infty$. Moreover, the parameter q determines the behavior of the lagrangian density $f(X)$ around the values of $X = -\phi^{'2}(r)$ in the limits of the integral. This allows to obtain supplementary conditions to be imposed on this function in order to have finite-energy SSS solutions. In this way the discussion of the different possible behaviors of the solutions at $r \sim 0$ and as $r \to \infty$ leads to a classification of all admissible models supporting this kind of solutions (see reference [11] for more details). When $r \sim 0$ we can distinguish three subcases (see Fig.1). If $0 > q > -1$ (case A-1) the lagrangian density shows a vertical parabolic branch as $X \to -\infty$ and the field strength of the SSS solution diverges at the origin, but the integral of energy converges there and the field potential $\phi(r)$ remains finite. If $q = 0$ (case A-2) the field strength of the SSS solution remains finite at the origin ($\phi^{'}(r \sim 0) \sim C - \beta r^\sigma$), and the slope of the lagrangian density diverges at $X = -\phi^{'2}(0) = -C^2$. The lagrangian $f(X)$ itself can take a negative finite value at this point (for $\sigma > 2$; the scalar version of the BI model corresponds to $\sigma = 4$) or show a vertical asymptote there (for $\sigma \leq 2$). The case $q > 0$ (case A-3) must be discarded. Indeed, if $2 \geq q > 0$ the lagrangian density diverges in vacuum and if $q > 2$ the energy density becomes negative around $X = 0$.

Concerning the asymptotic behavior of the solutions we can also distinguish three subcases (see Fig.1). When $-2 < q < -1$ (case B-1) the slope of the lagrangian density vanishes around $X \sim 0$. The definiteness of the lagrangian around the origin restricts the admissible values of q to the rational numbers, by means of the relation $q = \frac{4+2\Sigma}{\Sigma-4}$, where $\Sigma = N_1/N_2$ is the irreducible ratio of two odd naturals such that $N_1 < N_2$. For $q = -2$ (case B-2) the lagrangian density behaves as the D'Alambert lagrangian around $X \sim 0$ ($f(X \to 0^\pm) \sim X$) and the field strength is asymptotically coulombian. When $q < -2$

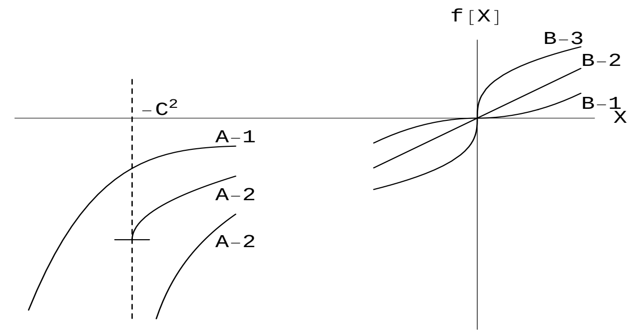

FIG. 1: Different possible behaviors of the admissible lagrangians with finite-energy static central field solutions.

(case B-3) the slope of the lagrangian diverges at $X \sim 0$, but the lagrangian itself remains well defined there if the admissible values of q are restricted to the set of rational numbers through the formula $q = 2\frac{\Sigma+1}{\Sigma-1}$, where now $\Sigma = N_1/N_2$ is the irreducible ratio between an even natural N_1 and an odd natural $N_2 > N_1$.

Let us give three illustrative examples of admissible models belonging to the different cases and showing soliton solutions (see Ref.[11] for a more detailed analysis of these models):

$$\begin{aligned} L_1 &= \frac{X}{2} + \lambda X^a \ ; \ \lambda > 0, \ a = \frac{odd}{odd} > \frac{3}{2} \\ L_2 &= \frac{(1+\mu^2 X)^\alpha - 1}{2\alpha\mu^2} \ ; \ 1/2 \leq \alpha < 1 \\ L_3 &= \frac{X^\alpha}{2(1+\mu^2 X)^a}; 0 < a = \frac{odd}{even} < 1; \alpha = \frac{odd}{odd} > a + \frac{1}{2} \end{aligned} \quad (7)$$

The first lagrangian (L_1) defines a two-parameter family of class-1 field theories which fall into the cases A-1 and B-2 (field strength with an integrable singularity at the center and coulombian asymptotic behavior). The constant λ gives the intensity of the self-coupling and the family reduces to the D'Alambert model as $\lambda \to 0$. It can be generalized to larger families of models with soliton solutions, including lagrangians which are odd-powers series expansions in X or odd functions of X. The second lagrangian (L_2) defines a two-parameter family of class-2 field theories, belonging to the cases A-2 (finite limit of $f(X)$ at $X = -\phi^{'2}(0)$ and field strength finite at the center) and B-2 (coulombian asymptotic behavior). The value $\alpha = 1/2$ in this family corresponds to the scalar version of the BI model. As the parameters $\mu \to 0$ or $\alpha \to 1$ all the models converge to the D'Alambert one, and the SSS solutions to the Coulomb field. The third lagrangian (L_3) defines a three-parameter family of class-2 field theories belonging to the cases A-2 (showing a vertical asymptote at $X = -\phi^{'2}(0)$) and to the cases B-1, B-2 or B-3, depending on $\alpha \gtreqless 1$. In all cases the strength of the field is finite at the origin ($\phi^{'}(0) = 1/\mu$) and behaves like $r^{-2/(2\alpha-1)}$ asymptotically.

Let us now address the question of the stability of the SSS finite-energy solutions. The linear stability of these solutions requires their energy to be a minimum against small perturbations preserving the scalar charge of the soliton. At the first order the (vanishing) modification of this charge by a small perturbation $\delta\phi(\vec{r})$, assumed to be finite and regular everywhere, takes the form

$$\begin{aligned} \Delta\Lambda = \int d_3\vec{r}\vec{\nabla}.[\dot{f}(X_0)(\vec{\nabla}\delta\phi) - \\ -2\ddot{f}(X_0)(\vec{\nabla}\phi.\vec{\nabla}\delta\phi)\vec{\nabla}\phi] = 0, \end{aligned} \quad (8)$$

where now $X_0 = -\phi^{'2}(r)$. The condition $\Delta\Lambda = 0$ imposes restrictions on the behavior of the admissible perturbations at $r = 0$ and as $r \to \infty$. In particular, $\delta\phi$ must vanish asymptotically faster than the solution ϕ approaches its asymptotic value. In this way the perturbed fields remain inside the space of functions defined by the prescribed boundary conditions which determine uniquely the solutions. Introducing the perturbed function in the integral of Eq.(3) and expanding up to the second order, it is easily seen that the first variation of the energy vanishes. This is the necessary condition for the soliton energy to be an extremum. The second variation reads

$$\Delta_2\varepsilon = \int d_3\vec{r}\left[\dot{f}(X_0)(\vec{\nabla}\delta\phi)^2 - 2\ddot{f}(X_0)(\vec{\nabla}\phi.\vec{\nabla}\delta\phi)^2\right], \quad (9)$$

which, owing to Eq.(8), converges for any charge-preserving perturbation. The positivity of $\Delta_2\varepsilon$, which is the sufficient condition for linear stability, requires

$$\dot{f}(X_0) + 2X_0\ddot{f}(X_0) = \frac{-2\Lambda}{r^3\phi''(r)} > 0, \quad (10)$$

to be satisfied in all the range of values of $X = X_0$ covered by the solution (the equality is obtained by deriving Eq.(5) with respect to r). This condition is always fulfilled by admissible models with finite-energy SSS solutions, owing to the monotonic character of $\phi^{'}(r)$. We conclude that all finite-energy SSS solutions of admissible models are linearly stable against charge-preserving perturbations. We can perform a more detailed analysis of the dynamics of the small perturbations ($\delta\phi(\vec{r},t)$) governed by the linear equation

$$\begin{aligned} \vec{\nabla}.\left[\dot{f}(X_0)\vec{\nabla}(\delta\phi) - 2\ddot{f}(X_0)(\vec{\nabla}\phi.\vec{\nabla}\delta\phi)\vec{\nabla}\phi\right] - \\ -\frac{\partial}{\partial t}\left(\dot{f}(X_0)\frac{\partial\delta\phi}{\partial t}\right) = 0, \end{aligned} \quad (11)$$

obtained from the linearization of Eq.(2) around the SSS solutions. Note that (11) also takes the form of a conservation law. The general conditions imposed on the admissible models, besides the knowledge of the behavior of the SSS solutions around $r = 0$ and as $r \to \infty$, allow to perform the standard spectral analysis of these linear problems, without explicit specification of the particular form of the lagrangians [11]. We summarize here the main conclusions of this analysis, which can be deduced from the separation of space and time variables in Eq.(11) together with the boundary condition (8). There is, in all cases, a discrete spectrum of eigenvalues, whose associated eigenfunctions are mutually orthogonal and finite-norm, with respect to the scalar product defined as the spatial integral of the products of the functions with $\dot{f}(X_0(r))$ as kernel. The corresponding time dependence is oscillatory. Any initial perturbation, bounded in the norm, remains bounded as time evolves, confirming the linear stability.

All these results can be generalized to the case of N-components scalar fields with a dynamics governed by

lagrangian density functions of the form

$$L(\phi_i, \partial_\mu \phi_i) = f\left(\sum_{i=1}^{N} \partial_\mu \phi_i . \partial^\mu \phi_i\right), \qquad (12)$$

where, as in the scalar case, $f(X)$ is a given *continuous, derivable and monotonically increasing* function. As already mentioned, the analysis of this problem is also a necessary step in the generalization of these methods to non-abelian gauge field theories. The field equations associated with the lagrangians (12) take now the form of N local conservation laws:

$$\partial_\mu \left(\dot{f}(X) \partial^\mu \phi_i\right) = 0, \qquad (13)$$

where $X = \sum_{i=1}^{N} \partial_\alpha \phi_i . \partial^\alpha \phi_i$. For the SSS solutions $\phi_i(r)$, these equations have N first integrals of the form

$$r^2 \phi_i^{'} \dot{f}\left(-\sum_{j=1}^{N} \phi_j^{'2}\right) = \Lambda_i, \qquad (14)$$

where $\phi_i^{'} = d\phi_i/dr$ and Λ_i are integration constants. The canonical energy density reads

$$\rho(x) = 2\dot{f}(X) \sum_{i=1}^{N} \left(\frac{\partial \phi_i}{\partial t}\right)^2 - f(X), \qquad (15)$$

and is positive definite under the same conditions constraining the function $f(X)$ in the one-component case (in fact the set of conditions (4) defining "admissibility" are assumed to hold also in this case). In order to solve the system (14) we introduce the functions $X_i(r) = -\phi_i^{'2}(r)$ and $X(r) = \sum_{i=1}^{N} X_i(r)$. By squaring and adding Eqs.(14) we are lead to

$$r^2 \sqrt{-X} \dot{f}(X) = \Lambda, \qquad (16)$$

where $\Lambda = \sqrt{\sum_{i=1}^{N} \Lambda_i^2}$. We see that this equation is formally identical to Eq.(5) and, if the function $f(X)$ is the same in both cases, there is a one to one correspondence between the solutions of the scalar case $\phi^{'}(r, \Lambda)$ and the spheres S_Λ of radius Λ in the N-dimensional Λ_i-space, associated with sequences of solutions of the multiscalar case of the form

$$\phi_i^{'}(r, \Lambda_j) = \frac{\Lambda_i}{\Lambda} \phi^{'}(r, \Lambda). \qquad (17)$$

The constants Λ_i can now be identified as "charges" associated with the different components of the SSS field, in analogy with the BI and the one-component scalar cases. The energy of these solutions, obtained from the integration of (15), is the same as the energy of their scalar counterparts, obtained from (6). There is a degeneracy in S_Λ, obviously related to the rotational symmetry in the internal space of the lagrangian (12). Moreover, the conditions determining the multiscalar field models with finite-energy SSS solutions coincide with those already discussed in the scalar case. Concerning the conditions for stability of the solutions (17), the analysis of the one-component case can be straightforwardly generalized to the present situation [11]. The final conclusion is that the multicomponent SSS solitons are linearly stable against any perturbation preserving the scalar charges Λ_i, if the associated one component solitons are linearly stable (note also that transitions in the degeneration sphere S_Λ are blocked by the charge conservation conditions).

Some additional comments on the general stability of these *non-topological* solitons are in order. Obviously, their linear stability does not guarantee the conservation (or even a proper definition) of the "soliton identity" in presence of strong "external" fields. As is well known, in many examples of *topological* solitons their presence in any field configuration can be detected through the existence of associated discrete topological charges, which are conserved no matter the intensity of external interactions. In some few cases (always in one-space dimension) explicit exact many-soliton solutions have been found, allowing the direct analysis of the dynamics of the system in terms of interacting solitons [12]. But a satisfactory *general* analysis of the interaction of solitons with strong external fields in three space dimensions is still lacking. However, there are some tentative approaches to this question which have been developed in the framework of the Born-Infeld model. We mention the method advanced in Ref. [13], based on the use of the discontinuity of the field strength at the center of *static* BI solitons, as a marker of the presence and location of the *dynamic* soliton in strong external fields. Since all our soliton solutions exhibit similar central singularities, this procedure could be generalized to the models considered here. But, in any case, the permanence of these solitons in strong interactions remains a hypothesis which, at best, is compatible with this method. On the other hand, in the case of weak external fields linear stability *does* imply identity preservation. Consequently, linear stability is a basic condition for the consistency of *low-energy* calculations of the interaction between solitons and weak fields (or between distant solitons). The results of such calculations may be interpreted in terms of particle-field (or particle-particle) force laws and give also a first approach to the radiative behavior in these processes [6].

As already mentioned, the present analysis has been generalized to electromagnetic and non-abelian gauge fields of compact semisimple Lie groups [10]. The main result amounts to establish a correspondence between any scalar model with stable SSS solutions of finite energy and families of gauge field models exhibiting similar solutions (which can be explicitly written in terms of the scalar ones). Conversely, the families so defined exhaust the class of gauge field models supporting the aforementioned kind of solitons.

Acknowledgements

We are grateful to Dr. A.R. Plastino for a careful reading of the manuscript and very helpful comments.

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[14] This assumption excludes transcendent asymptotic behavior of the field strength as, for example, exponential damping at infinity, but our conclusions will not be affected by this restriction. In fact such models belong to the case B-3 defined below.